

SOLUTION OF THE VACUUM KERR-SCHILD PROBLEM[†]

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ABSTRACT

The complete solution of the vacuum Kerr-Schild equations in general relativity is presented, including the space-times with a curved background metric. The corresponding result for a flat background has been obtained by Kerr.

The Kerr-Schild pencils¹ of metrics

$$\tilde{g}_{ab} = g_{ab} + V l_a l_b \quad (1)$$

have been in the forefront of research in general relativity for some time. A classic example of these space-times is the Kerr metric. All the Kerr-Schild vacuum space-times with a flat parent metric g_{ab} are given by beautiful geometrical relations². The solution of the flat problem (known as *Kerr's theorem*) reveals a close relationship with complex surfaces in three-dimensional homogeneous spaces. Kerr-Schild congruences in Minkowski space-time have extensively been studied^{3–6}. Work on Kerr-Schild space-times in the generic case when g_{ab} has a nonvanishing curvature, l is a null vector and V a function, has certainly been motivated by the prospects of extending Kerr's complex analytic description to curved space-times^{7–10}.

In this Letter we present the complete solution of the vacuum Kerr-Schild problem. The Kerr-Schild equations follow from (1) and the vacuum Einstein equations. They are:

$$Dl_a = 0 \ , \quad (2)$$

$$\nabla_b [\nabla_a (V l_c l^b) + \nabla_c (V l_a l^b) - \nabla^b (V l_a l_c)] = 0 \quad (3)$$

$$DDV + (\nabla^a l_a)DV + 2V(\nabla_b l_d)\nabla^{[b}l^{d]} = 0. \quad (4)$$

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Here ∇ is the covariant derivative annihilating g_{ab} . The vector

$$D = l^a \nabla_a = \partial/\partial r \quad (5)$$

is tangent to a null geodesic congruence⁸, with r the affine parameter.

We rewrite these equations in a Newman-Penrose (NP) notation¹², choosing l a vector of the null tetrad. The geodesic condition (2) becomes $\kappa = 0$. We adopt the gauge with $\epsilon = 0$, $\pi = \alpha + \bar{\beta}$, and integrate a closed subset of the ‘radial’ field equations¹² for the affine parameter dependence:

$$\rho = -\frac{1}{2r}(1 + \cos \eta \, C) \, , \quad \sigma = -\frac{\sin \eta}{2rC} \quad (6)$$

$$\Psi_0 = -\frac{\sin 2\eta}{4r^2} \, . \quad (7)$$

Here C is a complex phase factor

$$C = \frac{r^{\cos \eta} - iB}{r^{\cos \eta} + iB} \, . \quad (8)$$

B and η are real integration functions. A further integration function has been eliminated by suitably fixing the origin of the affine parameter. The real potential can be written as

$$V = V_0 \frac{r^{\cos \eta}}{r^{2 \cos \eta} + B^2} \, . \quad (9)$$

The complex tetrad vector m has the r dependence

$$m = \frac{1}{2B} \left(1 - \frac{1}{C}\right) \left[iQ_1^j r^{\frac{\cos \eta - \sin \eta - 1}{2}} - Q_2^j r^{\frac{\cos \eta + \sin \eta - 1}{2}} \right] \frac{\partial}{\partial x^j} \quad j = 1, 2, 3 \quad (10)$$

where V_0, Q_1^j and Q_2^j are real integration functions. With this choice, the tetrad has been uniquely fixed.

The real function B controls the amount of divergence and rotation of the null congruence with tangent l , respectively. B gives rise to the imaginary part of the spin coefficient ρ . When $B = 0$, we have $C = 1$, and the congruence is curl-free. Similarly, for large values of the affine parameter r , the phase factor C approaches the unit value, and the rotation dies out. The parameter η governs the shear. For $\eta = 0$ or $\eta = 180^\circ$, the congruence is shear-free. When both $B = 0$ and $\eta = 0$, the rays are exactly spherical:

$\rho = -1/r$. For $\eta = 90^\circ$, the rays become cylindrical, $\rho = -1/2r$. At $\eta = 180^\circ$, there is no expansion. Our Theorem below shows that the general shearing class does not contain the shear-free case as a smooth limit.

The field equations in the NP form may be grouped in three sets. The first set of equations is a coupled system of linear inhomogeneous equations for the affine parameter dependence of the quantities $\tau, \pi, \alpha, \beta, \Psi_1$ and for their complex conjugates:

$$D\tau = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \Psi_1 \quad (11a)$$

$$THE \quad D\pi = 2\rho\pi + 2\bar{\sigma}\bar{\pi} + \bar{\Psi}_1 \quad (11b)$$

$$\Psi_1 \quad D\alpha = \rho(\pi + \alpha) + \bar{\sigma}(\bar{\pi} - \bar{\alpha}) \quad (11c)$$

$$EQS. \quad D\beta = \bar{\rho}\beta + \sigma(2\pi - \bar{\beta}) + \Psi_1 \quad (11d)$$

$$D\Psi_1 = 4\rho\Psi_1 + (\bar{\delta} + \pi - 4\alpha)\Psi_0. \quad (11e)$$

The Ψ_1 equations are complemented by two equations, linear in the unknown functions:

$$\delta\left(\frac{\Psi_0}{2\sigma}\right) + \frac{\Psi_0}{\sigma}\delta\ln\phi - 2\sigma\bar{\delta}\ln\phi - \Psi_1 = \frac{\Psi_0}{2\sigma}(\tau - \bar{\alpha} - \beta) - 2\sigma(\bar{\tau} - \alpha - \bar{\beta}) + \bar{\tau}\sigma - \tau\rho \quad (12)$$

$$\delta\rho - \bar{\delta}\sigma = \rho\bar{\pi} + \sigma(\pi - 4\alpha) + (\rho - \bar{\rho})\tau - \Psi_1. \quad (13)$$

where $\phi = \sqrt{V}$.

Theorem: *For a generalized vacuum-vacuum Kerr-Schild pencil, either of the following conditions holds: The parameter η assumes one of the special values given by*

$$\sin\eta = 0, \quad \pm 1, \quad \pm 2^{-\frac{1}{2}} \quad . \quad (14)$$

or, alternatively, the spin coefficient quantities ρ, σ and Ψ_0 depend only on the affine parameter r .

The theorem can be proven¹¹ by taking the D derivative of Eq. (12), and eliminating the unknown spin coefficients by use of the NP commutators, Eq. (13) and the equations of the Ψ_1 system. This yields that $\delta\eta = 0$ and, unless η takes any of the exceptional values, the integration functions are restricted by $\delta V_0 = \delta B = 0$. In the generic case, $\rho - \bar{\rho} \neq 0$, the commutator equations imply also $\Delta\phi = 0$ for any real function ϕ with $D\phi = \delta\phi = 0$. Hence the integration functions B, η and V_0 are constants. The curl-free fields, $B = 0$, will be discussed in Ref. 11.

It follows that the Kerr solution with $\eta = 0$ will *not* emerge as a smooth limiting case of the shearing Kerr-Schild metrics.

For fields of arbitrary deformation parameter, α and Ψ_1 may be expressed algebraically in terms of π, τ and their complex conjugates, by the respective Eqs. (12) and (13). The Ψ_1 system becomes a quartet of coupled equations for the spin coefficients τ, π and their complex conjugates:

$$\begin{aligned} D\tau &= (2\rho - \frac{\Psi_0}{2\sigma})\tau + \sigma(2\bar{\tau} - \pi) + (\rho + \frac{\Psi_0}{2\sigma})\bar{\pi} \\ D\pi &= (2\rho + \frac{\Psi_0}{2\sigma})\pi + \bar{\sigma}\tau + (\bar{\rho} - \frac{\Psi_0}{2\bar{\sigma}})\bar{\tau} \end{aligned} \quad (15)$$

along with the complex conjugate equations. The fundamental solution of the Ψ_1 set is given in Table 1. The general solution is a linear combination of the four fundamental solution vectors with *real* coefficients.

The radial component of the vector n may be obtained by applying the commutator $[\delta, \bar{\delta}]$ on r . Hence the operator Δ , when acting on any of the functions depending only on r , takes the form $\Delta = -\frac{\bar{\mu}-\mu}{\bar{\rho}-\rho}D$. The second set of field equations consists of the NP Eqs. (4.2l,p,q), as well as the fifth of NP (4.5) and of the Kerr-Schild equation

$$\begin{aligned} &\left[\frac{1}{2}\frac{\bar{\rho}+\rho}{\bar{\rho}-\rho}\left(\frac{1}{r} + \bar{\rho} + \rho\right)\right](\bar{\mu}-\mu) + \frac{1}{2}\left(\frac{\Psi_0}{\sigma} + \rho - \bar{\rho}\right)(\bar{\mu}+\mu) + \Psi_2 + \bar{\Psi}_2 - (\rho + \bar{\rho})(\gamma + \bar{\gamma}) \\ &= \delta(\bar{\tau} - \pi) + \bar{\delta}(\tau - \bar{\pi}) - 6\pi\bar{\pi} + 2\pi\bar{\alpha} + 2\bar{\pi}\alpha - 2\tau\bar{\tau} - 2\tau\alpha - 2\bar{\tau}\bar{\alpha} + 3\tau\pi + 3\bar{\tau}\bar{\pi} . \end{aligned} \quad (16)$$

We obtain a lengthy relation from these, containing only functions the r -dependence of which is explicitly known. Collecting the coefficients of independent powers of r , one finds that nothing but the trivial solution of the Ψ_1 system satisfies this equation:

$$\alpha = \beta = \tau = \pi = \Psi_1 = 0 . \quad (17)$$

Thus the second system of equations is homogeneous and linear in μ, λ, γ and Ψ_2 . The determinant vanishes, and we get

$$\frac{\gamma}{\rho + 1/2r} = \frac{\lambda}{\bar{\sigma}} = -\frac{\Psi_2\sigma}{\Psi_0\rho} = \frac{\mu}{\rho} . \quad (18)$$

The remaining field equations constitute the third set. They further restrict the metric, leaving us with a three-parameter pencil for which the image of the Kerr-Schild map is the Kóta-Perjés¹³ metric (44):

$$ds^2 = -\frac{r^{\cos 2\eta} + B^2}{r^{\cos \eta}}(dr^2 + r^{1-\sin \eta}dx^2 + r^{1+\sin \eta}dy^2) + \Lambda\frac{r^{\cos \eta}}{r^{\cos 2\eta} + B^2}(l_a dx^a)^2 . \quad (19)$$

Here $l = \partial/\partial r$ is tangent to the Kerr-Schild congruence and Λ is the pencil parameter.

To complete our investigation of vacuum Kerr-Schild space-times, we consider now in turn the metrics with either of the values (14).

(a) When $\sin \eta = 0$, both σ and Ψ_0 vanish, and l is a principal null vector of the curvature. By the Goldberg-Sachs theorem¹², these parent space-times are algebraically special, and $\Psi_1 = 0$. It then follows from Thompson's Theorem 3.2 that also the ensuing space-time is algebraically special, with the Kerr-Schild congruence being a principal null congruence⁸. All the vacuum Kerr-Schild spacetimes generated from the flat space-time are in this class.

(b) The case with $\cos \eta = 0$ maps Minkowski space-time to itself.

(c) Case $\sin \eta = 1/\sqrt{2} = k$ contains the following Kóta-Perjés metrics:

$$ds^2 = -\frac{f^0}{f}(r^{1-k}dx^2 + r^{1+k}dy^2) + 2dr(l_a dx^a) + f(l_a dx^a)^2 \quad (20) .$$

Metric (53) of Ref. 8 is given by

$$f = \Lambda \operatorname{Re} \left\{ \frac{x + ir^k y}{r^k + iB} \right\}, \quad f^0 = \Lambda(x + By) \quad (21)$$

with B a real constant. For metric (66), $B = x/y$ and

$$f = \Lambda \frac{x + by}{x^2 r^{-k} + y^2 r^k}, \quad f^0 = \Lambda(x + by)/y^2 . \quad (22)$$

Notice that we have enlisted above all the metrics in Ref. 8.

We thus find that all Kóta-Perjés metrics are explicit cases of Kerr-Schild space-times, either with a real deformation parameter or with $\eta = 1/\sqrt{2}$.

In summary, the structure of our solution is as follows. The Kerr-Schild space-times are characterized by the real deformation parameter η . The deformation parameter vanishes for Kerr-Schild space-times the parent of which is Minkowski space-time. By our theorem, the field quantities are severely restricted unless the parameter η assumes either of the exceptional values $0, \pm 1, \pm \sqrt{2}/2$. The metrics with arbitrary values of the deformation parameter are Kóta-Perjés metrics. For the exceptional values of η , (a)

the class with $\eta = 0$ is algebraically special, (b) the values $\sin \eta = \pm 1$ can occur only in automorphisms of the Minkowski space-time, and (c) the class with $\sin \eta = \pm 1/\sqrt{2}$ contains the remaining Kóta-Perjés metrics. While the Kerr-Schild congruences in a Minkowski space-time form a four-parameter family², there is no corresponding structure on a curved background. Our results have the important implication that hopes are dashed for a complex-analytic description of space-time within the framework of Kerr-Schild theory.

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$$\pi^{(1)} = \frac{C+1}{r^{\frac{\cos \eta + \sin \eta + 1}{2}}} \left(C^{-3} - \frac{\cos \eta}{\sin \eta + 1} C^{-2} - \frac{\sin \eta}{\sin \eta + 1} C^{-1} \right. \\ \left. + \frac{5 \sin^2 \eta - 4 \sin \eta + 3}{\cos \eta (\sin \eta - 3)} + \frac{3 \sin \eta - 1}{\sin \eta + 1} \frac{2 \sin \eta + 3}{\sin \eta - 3} C + 2 \frac{\sin \eta}{\cos \eta} C^2 \right)$$

$$\pi^{(2)} = i \frac{C+1}{r^{\frac{\cos \eta - \sin \eta + 1}{2}}} \left(C^{-3} + \frac{\cos \eta}{\sin \eta - 1} C^{-2} - \frac{\sin \eta}{\sin \eta - 1} C^{-1} \right. \\ \left. + \frac{5 \cos^2 \eta - 4 \sin \eta - 8}{\cos \eta (\sin \eta + 3)} + \frac{6 \cos^2 \eta + 7 \sin \eta - 3}{\cos^2 \eta - 2 \sin \eta + 2} C - 2 \frac{\sin \eta}{\cos \eta} C^2 \right)$$

$$\pi^{(3)} = i \frac{C-1}{r^{\frac{\cos \eta + \sin \eta + 3}{2}}} (C+1)^2 \left(-C^{-3} - \frac{\sin \eta + 1}{\cos \eta} C^{-2} + \frac{1}{\sin \eta - 1} C^{-1} - 2 \frac{\sin \eta}{\cos \eta} \right)$$

$$\pi^{(4)} = \frac{C-1}{r^{\frac{\cos \eta - \sin \eta + 3}{2}}} (C+1)^2 \left(-C^{-3} - \frac{\cos \eta}{\sin \eta + 1} C^{-2} - \frac{1}{\sin \eta + 1} C^{-1} + 2 \frac{\sin \eta}{\cos \eta} \right)$$

$$\tau^{(1)} = \frac{C+1}{r^{\frac{\cos \eta + \sin \eta + 1}{2}}} \left(\frac{\sin \eta}{\cos \eta} C^{-3} - \frac{\sin \eta}{\sin \eta + 1} C^{-2} - \frac{\sin^2 \eta + 8 \sin \eta + 3}{\cos \eta (\sin \eta - 3)} \frac{\sin \eta - 1}{\sin \eta + 1} C^{-1} \right. \\ \left. - 3 \frac{\sin \eta}{\sin \eta + 1} + \frac{2 \sin \eta - 3}{\cos \eta} C + 2 C^2 \right)$$

$$\tau^{(2)} = i \frac{C+1}{r^{\frac{\cos \eta - \sin \eta + 1}{2}}} \left(\frac{\sin \eta}{\cos \eta} C^{-3} + \frac{\sin \eta}{\sin \eta - 1} C^{-2} - \frac{\sin \eta \cos^2 \eta - 7 \cos^2 \eta + 4 \sin \eta + 4}{\cos \eta (2 \sin \eta - 2 - \cos^2 \eta)} C^{-1} \right. \\ \left. + 3 \frac{\sin \eta}{\sin \eta - 1} + \frac{2 \sin \eta + 3}{\cos \eta} C - 2 C^2 \right)$$

$$\tau^{(3)} = i \frac{C-1}{r^{\frac{\cos \eta + \sin \eta + 3}{2}}} (C+1)^2 \left(-\frac{\sin \eta}{\cos \eta} C^{-3} + \frac{\sin \eta}{\sin \eta - 1} C^{-2} - \frac{2 \sin \eta + 1}{\cos \eta} C^{-1} - 2 \right)$$

$$\tau^{(4)} = \frac{C-1}{r^{\frac{\cos \eta - \sin \eta + 3}{2}}} (C+1)^2 \left(-\frac{\sin \eta}{\cos \eta} C^{-3} - \frac{\sin \eta}{\sin \eta + 1} C^{-2} - \frac{2 \sin \eta - 1}{\cos \eta} C^{-1} + 2 \right) .$$

Table 1. *The four solutions for π and τ*